

Boundary correlation numbers in one matrix model

Goro ISHIKI^{1*} AND Chaiho RIM^{2†}

*Department of Physics² and Center for Quantum Spacetime (CQUeST)^{1,2}
Sogang University, Seoul 121-742, Korea*

Abstract

We introduce one matrix model coupled to multi-flavor vectors. The two-flavor vector model is demonstrated to reproduce the two-point correlation numbers of boundary primary fields of two dimensional $(2, 2p+1)$ minimal Liouville gravity on disk, generalizing the loop operator (resolvent) description. The model can properly describe non-trivial boundary conditions for the matter Cardy state as well as for the Liouville field. From this we propose that the n -flavor vector model will be suited for producing the boundary correlation numbers with n different boundary conditions on disk.

* e-mail address : ishiki@post.kek.jp

† e-mail address : rimpine@sogang.ac.kr

1 Introduction

There have been developed two independent approaches to (Euclidean) two dimensional quantum gravity since the middle of 80's: Liouville gravity (LG) [1–4] versus matrix models (MM) [5–10]. Two approaches are checked in a number of particular models [11–13]. One may refer to more references in reviews in [14, 15].

Initiated by Al. Zamolodchikov's direct computation of partition function on fluctuating sphere at the simplest integral point (Liouville coupling set to $b^2 = \frac{2}{5}$) [16], Belavin and Zamolodchikov succeeded to confirm the correlation numbers of minimal gravity by providing so-called resonance transformation formula between conformal (Liouville) and KdV frame [17].

Correlation numbers are given in terms of the integrated form of the local density (2-form operator) $\mathcal{O}_k(X)$, $O_k = \int_{\mathcal{M}} \mathcal{O}_k(X)$ over the manifold \mathcal{M} which accommodates both “matter” and the metric degrees of freedom localized at $X \in \mathcal{M}$. A generating function of the correlation numbers in Liouville gravity is given by $Z_{LG}(\{\lambda_k\}) = \langle e^{\sum_k \lambda_k O_k} \rangle$. A similar generating function, which depends on the parameters t_k , can be introduced in Matrix models. It was conjectured in [18] that there exists a special choice of contact terms in MM or, equivalently, the special transformation $t_k = t_k(\{\lambda_j\})$ such that the coincidence of the partition functions is ensured in the p -critical one-matrix model $OMM(p)$ and the Minimal Liouville gravity $MLG(2, 2p+1)$ for random surfaces with spherical topology. This relation between the parameters t_k and λ_k was obtained in [18] up to the linear terms. Its explicit form to all orders was conjectured in [17] and checked up to the 4-th order [17, 19].

Given the minimal gravity, it has been also shown [20] that the resonance transformation formula also works for bulk correlation numbers in the presence of fluctuating disk. In particular, the disk partition function of $OMM(p)$ with the boundary length l in the conformal frame is given as [18, 20]

$$\mathcal{Z}_D(\mu, l) = \frac{u_0^{p+1}}{\sqrt{l}} \int_1^\infty dx L_p(x) e^{-lu_0 x} = \sqrt{\frac{2}{\pi}} \frac{u_0^{p+1/2}}{l} K_{p+1/2}(u_0 l) \quad (1.1)$$

where μ is the bulk cosmological constant. On the other hand, the Liouville gravity partition function $Z_D(\mu, \mu_B)$ is related by the inverse Laplace transform of $\mathcal{Z}_D(\mu, l)$

$$\mathcal{Z}_D(\mu, l) = l \int_{\uparrow} \frac{d\mu_B}{2\pi i} e^{\mu_B l} Z_D(\mu, \mu_B), \quad (1.2)$$

where the contour \uparrow goes along the imaginary axis to the right from all the singularities of the integrand. Using the Laplace transform one can show that the resolvent in the continuum limit on the spherical topology

$$\omega(z) = \left\langle \text{tr} \left(\frac{1}{z - M} \right) \right\rangle = \left\langle \int_0^\infty dl \text{tr} e^{-l(z-M)} \right\rangle \quad (1.3)$$

is proportional to $\cosh(\pi s/b)$ when the boundary parameter is set to $z = u_0 \cosh(\pi b s)$. u_0 provides a scale parameter in the matrix model (details are given in appendix A). For p -critical theory, $\omega(z)$ has the scale dimension $u_0^{p+1/2}$ so that $\omega(z) = u_0^{p+1/2} \cosh(\pi s/b)$.

In Liouville field theory (LFT), b corresponds to the Liouville coupling constant relating the background charge $Q = b + 1/b$ and z to the boundary cosmological constant $\mu_B = \sqrt{\mu/\sin(\pi b^2)} \cosh(\pi b s)$. $\text{MLG}(2, 2p+1)$ corresponds to the case $b^2 = 2/(2p+1)$. Boundary operator of LFT, $^{s_1} [e^{\beta\phi}(X)]^{s_2}$ with $X \in \partial\mathcal{M}$ has conformal dimension $\Delta(\beta) = \beta(Q - \beta)$ and is specified by the boundary condition $\text{BC}(s_1; s_2)$ which $e^{\beta\phi}$ joins. The primary Liouville boundary operator in $\text{MLG}(2, 2p+1)$ is given as $B_{1\ell} = e^{\beta_{1\ell}\phi} \Phi_{1\ell}$ and $\beta_{1\ell} = b(1 + \ell)/2$ with $\ell = 1, 2, \dots, p$. Φ_{mn} represents the CFT matter in the Kac table with the conformal dimension $\Delta_{mn} = \alpha_{mn}(\alpha_{mn} - q)$ where $\alpha_{mn} = (n - 1)b/2 - (m - 1)/(2b)$ and $q = 1/b - b$. It should be noted that in LFT, the boundary correlation of $e^{\beta\phi}$'s is known in [21, 24], not to mention the boundary correlation for the A-series minimal model [22, 23]. However, the corresponding result is not available in the matrix model side, even though few attempts can be found in RSOS model and $O(N)$ fluctuation models [25, 26], in loop gas model [27, 28], and in (two) matrix model [29, 30].

In this paper, we propose boundary changing operator description in $\text{OMM}(p)$ which reproduces boundary correlation numbers of primary fields in $\text{MLG}(2, 2p+1)$ on disk. This paper is organized as follows. In section 2, we introduce one matrix model which couples to certain number of vectors. In section 3, we demonstrate that 2-point correlation number of the integrated form of boundary changing operator, $^{(s_1, 1)} \mathcal{B}_{1\ell}^{(s_2, \ell)} = \int_{\partial\mathcal{M}} ^{(s_1, 1)} B_{1\ell}^{(s_2, \ell)}(X)$ are reproduced from the matrix model with two vectors. Here s_1 and s_2 correspond to the boundary conditions of LFT and $1 = (1, 1)$ and $\ell = (1, \ell)$ are the Cardy label of minimal boundary conditions. Starting from $\text{BC}(s_1, 1; s_2, \ell)$ one can construct the general boundary condition, $\text{BC}(s_1, m; s_2, n)$, when $1 \leq m, n \leq p$ are allowed from fusion property. Section 4 is devoted to summary and discussion.

2 One matrix model with vectors

In order to describe 2 dimensional gravity with boundaries, we introduce one matrix model with vectors,

$$e^Z = \int DM Dv^{(a)\dagger} Dv^{(b)} \exp \left(-\frac{N}{g} \text{tr} V(M) - \sum_{a,b} v^{(a)\dagger} \cdot C^{(a,b)}(M) \cdot v^{(b)} \right), \quad (2.1)$$

where M is a $N \times N$ Hermitian matrix and $V(M)$ is a polynomial of M which starts from a quadratic term $\frac{1}{2}M^2$. $v^{(a)}$ and its hermitian conjugate $v^{(a)\dagger}$ are N dimensional vectors and \cdot represents the contraction of the N dimensional indices. a and b label the “flavors” of the vectors. The number of flavors may depend on the number of different boundary conditions. $C^{(a,b)}(M)$ is a hermitian matrix whose diagonal component is given as a polynomial of M ,

$$C^{(a,a)}(M) = \sum_{n=0}^{K^a} b_n^{(a)} M^{K^a-n}, \quad (2.2)$$

with $b_0^{(a)}$ normalized¹ as $(-1)^{K^a}$ and K^a is the order of the polynomial. $b_n^{(a)}$ in the diagonal component behaves as the source to the boundary preserving operator. The off-diagonal component is also given as a polynomial in general and its coefficients are the source to the boundary changing operator.

This model can generate Feynman diagrams which correspond to 2 dimensional discretized surfaces. We associate a double line with a propagator of M and a single line with that of the vectors as usual. The rule of drawing the Feynman diagram is shown in Fig. 1. A typical Feynman diagram with a boundary is found in Fig. 2. Note that the vector model (2.1) is quadratic in the vectors and thus, the propagators of the vectors always form loops which are regarded as boundaries.

The partition function Z in (2.1) can be expanded over various topologies including disks as following. We first integrate out the vectors and obtain

$$e^Z = \int DM \exp \left(-\frac{N}{g} \text{tr} V(M) - \text{Tr} \log C(M) \right) = e^{Z_0} \langle e^{-\text{Tr} \log C(M)} \rangle_0, \quad (2.3)$$

¹ Note that by rescaling the vectors, we can always fix the coefficient of the highest order term in M . Nevertheless, to compare the matrix result with the Liouville gravity one can choose a convenient normalization. More details are found in Sec. 3.

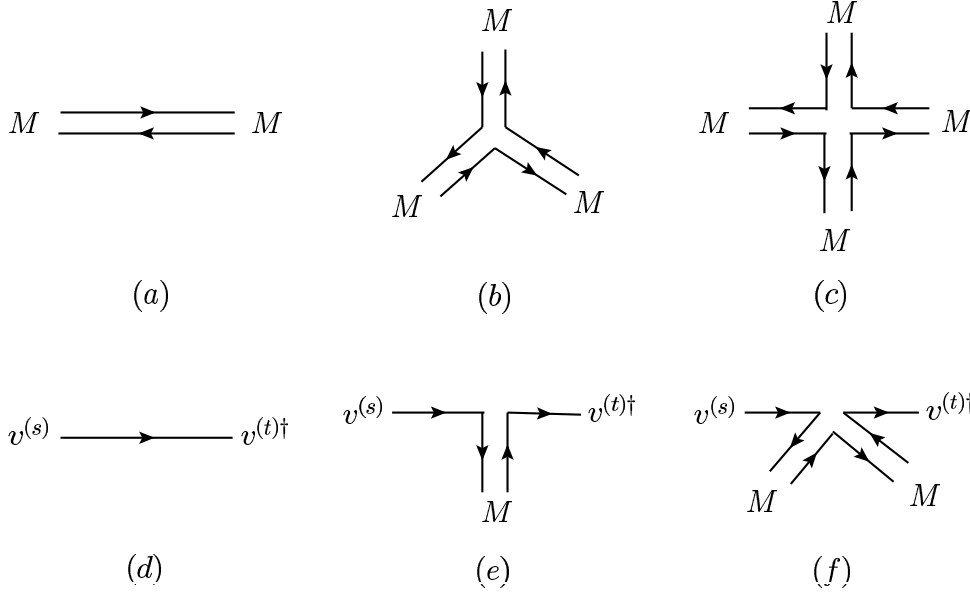


Figure 1: (a): Propagator for M . (b)(c): Examples of bulk interactions. (d): Propagator for the vectors. (e)(f): Examples of boundary interactions.

where Tr stands for the trace over both flavor and matrix indices. e^{Z_0} is the partition function without vectors

$$e^{Z_0} = \int DM \exp \left(-\frac{N}{g} \text{tr} V(M) \right) \quad (2.4)$$

and describes manifolds without any boundary. $\langle \cdots \rangle_0$ in (2.3) stands for a normalized expectation value with respect to (2.4). By taking the logarithm of (2.3), we obtain

$$Z = Z_0 + \langle e^{-\text{Tr} \log C(M)} \rangle_c \equiv Z_0 + \sum_{h=1}^{\infty} Z_h, \quad (2.5)$$

where $\langle \cdots \rangle_c$ denotes the connected part of $\langle \cdots \rangle_0$. Z_h becomes the partition function with $h \geq 1$ holes (boundaries)

$$Z_h = \frac{1}{h!} \langle (-\text{Tr} \log C(M))^h \rangle_c = \frac{1}{h!} \left\langle \left(\int_0^\infty \frac{dl}{l} \text{Tr} e^{-lC(M)} \right)^h \right\rangle_c. \quad (2.6)$$

Noting that an insertion of a traced operator lowers the order of N by 1 and thus, Z_h has the extra factor N^{-h} compared with Z_0 , we can topologically expand $Z_h = \sum_{g=0}^{\infty} Z_h^g N^\chi$, where $\chi = 2 - 2g - h$ is the Euler characteristic for the manifold with g handles and h holes. In the following, we restrict ourselves to the case of disk with no handle ($g = 0, h = 1$) for simplicity.

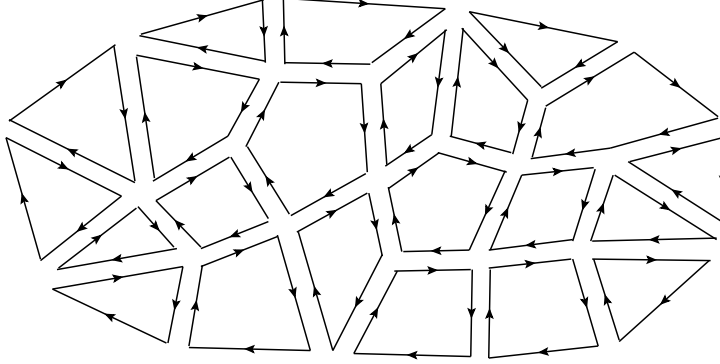


Figure 2: Typical Feynman diagram with a boundary.

3 Correlation numbers from one matrix model

Let us first consider the one-flavor case. Suppose $C(M) = x - M$. In this case, (2.6) with $h = 1$ reduces to the well-known simple partition function on disk,

$$Z_1 = -\langle \text{Tr} \log(x - M) \rangle_c = \left\langle \int_0^\infty \frac{dl}{l} \text{Tr} e^{-l(x-M)} \right\rangle_c. \quad (3.1)$$

The constant x couples to the length of boundary according to (1.2) and is identified as the boundary cosmological constant, which is real. Note that differentiating Z_1 with respect to x gives the resolvent $\omega(x)$.

Next, suppose the order of $C(M)$ is 2 ($K = 2$).

$$C(M) = c_2 + c_1 M + M^2 = (x^+ - M)(x^- - M). \quad (3.2)$$

c_1 and c_2 are real and $x^+ + x^- = -c_1$ and $x^+ x^- = c_2$. A certain correlation will be generated if one differentiates Z_1 with respect to the source c_2 :

$$O(x_1^+, x_1^-) \equiv \left\langle \text{tr} \frac{1}{(x^+ - M)(x^- - M)} \right\rangle = -\frac{w(x^+) - w(x^-)}{x^+ - x^-}. \quad (3.3)$$

When x^\pm are real, $O(x_1^+, x_1^-)$ is not vanishing and becomes the two point correlation number of the boundary cosmological operator \mathcal{B}_{11} [30]. However, if x^\pm is allowed complex this is not the whole story since $O(x_1^+, x_1^-)$ can vanish². In this case, $\partial Z_1 / \partial c_2$ can be interpreted as one point correlation of \mathcal{B}_{13} on disk.³ The vanishing condition is $\omega(x^+) =$

²This requirement is reminiscent of the method of solving the resonance relation on sphere [17].

³(3.3) is obtained from (3.7) when $x_1 \rightarrow \infty$ whose result is reproduced by fusing two \mathcal{B}_{12} 's. We thank Bourguine for this point.

$\omega(x^-)$, or $s^+ \pm s^- = 2ibn$ with n integer. In the following, the notations will be used $x^{(j)} = u_0 \cosh(\pi b s^j)$ and $w(x^{(j)}) = u_0^{p_0+1/2} \cosh(\pi s^j/b)$.

One can generalize the above argument to the case when $C(M)$ is the ℓ -th order polynomial of M . $C(M)$ is factorized in terms of ℓ solutions, $\{x^{(1)}, \dots, x^{(\ell)}\}$ satisfying $C(x^{(j)}) = 0$. Vanishing requirement on the one-point correlation $O(x^{(1)}, \dots, x^{(\ell)})$ gives a constraint between any of $x^{(j)}$'s so that $\omega(x^{(j)}) = \omega(x^{(k)})$ ($1 \leq j, k \leq \ell$);

$$\frac{s^j \pm s^k}{2ib} = 0 \pmod{1}. \quad (3.4)$$

Next, two different boundary conditions need two flavors. We put $C(M)$ as

$$C(M) = \begin{pmatrix} x_1 - M & c^{12} \\ c^{21} & F(M) \end{pmatrix} \quad (3.5)$$

where $F(M)$ is a certain polynomial of M and c^{21} is a M -independent constant and is the complex conjugate of c^{12} . Differentiating the partition function Z_1 with respect to the sources c^{12} and c^{21} one has two point correlation of boundary changing operators

$$\left. \frac{\partial^2 Z_1}{\partial c^{12} \partial c^{21}} \right|_{c^{ij}=0} = \left\langle \text{tr} \left(\frac{1}{x_1 - M} \frac{1}{F(M)} \right) \right\rangle. \quad (3.6)$$

When $F(M)$ is linear in M , the correlation trivially reduces to that of \mathcal{B}_{11} 's. To describe non-trivial ones, we may put $F(M) = (x_2^{(+)} - M)(x_2^{(-)} - M)$ and choose $s_2^\pm = s_2 \pm ib$ with s_2 real. This choice ensures $\omega(x_2^{(+)}) = \omega(x_2^{(-)}) = -\omega(x_2)$ with $x_2 = u_0 \cosh(\pi b s_2)$ and $C(M)$ hermitian. As the result, (3.6) becomes

$$\begin{aligned} O(x_1; \{x_2\}_2) &\equiv \left\langle \text{tr} \frac{1}{(x_1 - M)(x_2^{(+)} - M)(x_2^{(-)} - M)} \right\rangle \\ &= \frac{u_0^{p-\frac{3}{2}} \cosh\left(\frac{\pi s_p}{2b}\right) \cosh\left(\frac{\pi s_m}{2b}\right)}{2 \sinh\left(\frac{\pi b(s_p+ib)}{2}\right) \sinh\left(\frac{\pi b(s_p-ib)}{2}\right) \sinh\left(\frac{\pi b(s_m+ib)}{2}\right) \sinh\left(\frac{\pi b(s_m-ib)}{2}\right)}. \end{aligned} \quad (3.7)$$

where $s_p = s_1 + s_2$ and $s_m = s_1 - s_2$. This exactly agrees with Liouville two-point correlation of \mathcal{B}_{12} with $\text{BC}(s_1; s_2)$ for the Liouville field.

The generalization to other boundary operators is surprisingly simple: Put $F(M)$ as the ℓ -th order polynomial and choose (with s_2 real)

$$s_2^j = s_2 + ibj \quad \text{for } j = -(\ell-1), -(\ell-3), \dots, \ell-3, \ell-1, \quad (3.8)$$

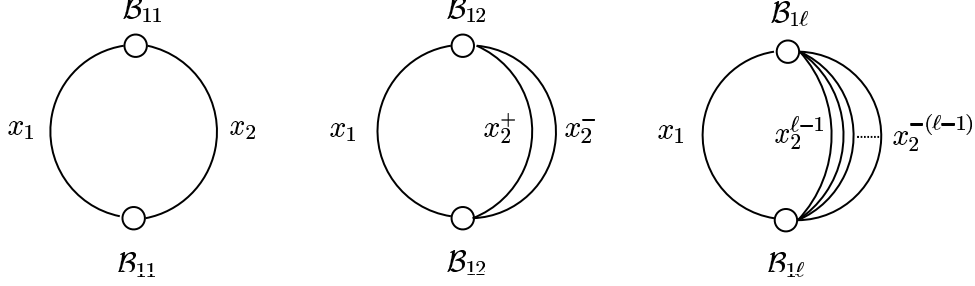


Figure 3: The boundary 2-point correlations for \mathcal{B}_{11} , \mathcal{B}_{12} and $\mathcal{B}_{1\ell}$. The line with x represents the factor of $\frac{1}{x-M}$.

so that $\omega(x_2^{(j)}) = \omega(x_2^{(j\pm 2)})$. Note that the number of allowed complex parameters for two point correlation is ℓ as shown in Fig. 3. Explicit evaluation of (3.6) is given as

$$O(x_1; \{x_2\}_\ell) \equiv \left\langle \text{tr} \left(\frac{1}{x_1 - M} \frac{1}{F_\ell(M; x_2)} \right) \right\rangle = (-)^\ell \frac{\omega(x_1) - \omega(x_2^{(\ell-1)})}{\prod_{k=0}^{\ell-1} (x_1 - x_2^{(\ell-1-2k)})} \quad (3.9)$$

where $F_\ell(M; x)$ is the polynomials of M with degree ℓ , $\prod_{k=0}^{\ell-1} (x^{(\ell-1-2k)} - M)$. This coincides with Liouville result of $s_1 \mathcal{B}_{1\ell}^{s_2}$ up to BC-independent normalization [21]. (See (A.16) of the first reference in [27] for the explicit formula for $b^2 = 2/(2p+1)$.)

One can also specify the boundary condition for the matter field in (3.9). Noting the boundary structure constant $c_{1\ell, 1\ell}^{(1, \ell, 1)} = 1$ [23], one concludes that Eq.(3.9) produces $\langle \mathcal{B}_{1\ell} \mathcal{B}_{1\ell} \rangle$ with $\text{BC}(s_1, 1; s_2, \ell)$ and $F_\ell(M; x)$ generates $\ell = (1, \ell)$ Cardy boundary condition. This conclusion is backed up by one-point correlation of \mathcal{B}_{11} with $\text{BC}(s, k)$,

$$O(\{x\}_k) = \frac{\partial}{\partial x} \text{tr} \log(F_k(M; x)) = [k]_q \omega(x) \quad (3.10)$$

where $[x]_q = (q^x - q^{-x})/(q - q^{-1})$ is the q -number with $q = \exp(i\pi b^2)$. From the field theoretic point of view, one can put the ratio $O(\{x\}_k)/O(\{x\}_1)$ as the ratio of the vacuum expectation value [23] of the matter part $\langle 1 \rangle_k / \langle 1 \rangle_1 = [k]_q$ since Liouville contribution cancels out in the ratio. The result is consistent with Eq. (3.10).

Incidentally, we note that the matter operator is identified as $\Phi_{1,k} = \Phi_{1,2p+1-k}$ and $\text{BC}(s; k) = \text{BC}(s; 1, 2p+1-k)$ and $\mathcal{B}_{1,k}$ is related to $\mathcal{B}_{1,2p+1-k}$ through the Liouville reflection (“unitary” condition) [21, 27]. Since the matrix result Eq. (3.9) shows the same functional dependence as the Liouville result, it goes without saying that the same Liouville reflection holds for the matrix result [27]. Thus, even though the boundary primary operator $\mathcal{B}_{1,k}$

is allowed as ($1 \leq k \leq 2p$) and the order of the diagonal component of $C(M)$ in Eq. (2.2) can be restricted to $1 \leq K^a \leq 2p$ for the p -critical model, the number of independent operators is further reduced by half by the Liouville reflection.

From the above consideration, we assert that $O(\{x_1\}_\ell; \{x_2\}_m)$ describes the two-point correlation of $\mathcal{B}_{1,\ell+m-1}$ with $\text{BC}(s_1, \ell; s_2, m)$. One can support this idea using the fusion rule. Let us consider the four-point correlation with simple BC ($\ell \leq m$ for definiteness)

$$\left\langle {}^{(s_1;1)}[\mathcal{B}_{1\ell}]^{(s_2;\ell)} {}^{(s_2;\ell)}[\mathcal{B}_{1\ell}]^{(s_3;1)} {}^{(s_3;1)}[\mathcal{B}_{1m}]^{(s_4;m)} {}^{(s_4;m)}[\mathcal{B}_{1m}]^{(s_1;1)} \right\rangle. \quad (3.11)$$

If one uses the operator fusion rule

$$\begin{aligned} {}^{(s_4;m)}[\mathcal{B}_{1m}]^{(s_1;1)} \otimes {}^{(s_1;1)}[\mathcal{B}_{1\ell}]^{(s_2;\ell)} &= \oplus_{|m-\ell+1| \leq k \leq m+\ell-1} {}^{(s_4;m)}[\mathcal{B}_{1k}]^{(s_2;\ell)} \\ {}^{(s_2;\ell)}[\mathcal{B}_{1\ell}]^{(s_3;1)} \otimes {}^{(s_3;1)}[\mathcal{B}_{1m}]^{(s_4;m)} &= \oplus_{|m-\ell+1| \leq k \leq m+\ell-1} {}^{(s_2;\ell)}[\mathcal{B}_{1k}]^{(s_4;m)} \end{aligned} \quad (3.12)$$

one ends up with the two point correlation with general BC's.

$$\left\langle {}^{(s_2;\ell)}[\mathcal{B}_{1k}]^{(s_4;m)} {}^{(s_4;m)}[\mathcal{B}_{1k}]^{(s_1;\ell)} \right\rangle. \quad (3.13)$$

From the matrix point of view, the four point correlation can be conjectured of the form

$$\left\langle \text{tr} \left(\frac{1}{F_1(M; x_1)} \frac{1}{F_\ell(M; x_2)} \frac{1}{F_1(M; x_3)} \frac{1}{F_m(M; x_4)} \right) \right\rangle. \quad (3.14)$$

Two point correlation is obtained by contracting the 1-boundary parts, which expands $1/F_1(M; x)$ in powers of M/x :⁴

$$\left\langle \text{tr} \left(M^g \frac{1}{F_\ell(M; x_2)} \frac{1}{F_m(M; x_4)} \right) \right\rangle \quad (3.15)$$

with $0 \leq g \leq \ell - 1$.⁵ This shows that the number of allowed correlations is the same as that of Eq. (3.13) and the range of k is $\ell + m - 1 - 2g \leq k \leq \ell + m - 1$. It is noted that (3.15) with $g \neq 0$ can be generated by $C(M)$ with M -dependent off-diagonal blocks,

$$C(M) = \begin{pmatrix} F_\ell(M; x_1) & G(M) \\ G(M)^\dagger & F_m(M; x_2) \end{pmatrix}. \quad (3.16)$$

When $g = 0$, Eq. (3.15) is indeed the 2-point correlation of $\mathcal{B}_{1,\ell+m-1}$ with $\text{BC}(s_1, \ell; s_2, m)$ up to a normalization. To elaborate on this, let us consider the two-flavor matrix model

⁴ M/x -expansion is equivalent to the small length expansion if one uses the Laplace transform Eq. (1.2).

⁵ When $g \geq \ell$, Eq. (3.15) reduces to the summation of Eq. (3.15)'s with $g < \ell$ except $\ell = m = 1$.

of the form (3.16) with $G(M)$ given by a M -independent constant: $G(M) = c^{12}$. For $\ell = k = 2$, one finds a relation after a short calculation,

$$\left. \frac{\partial^2 Z_1}{\partial c^{12} \partial c^{21}} \right|_{c^{ij}=0} = O(\{x_1\}_2; \{x_2\}_2) = 2 \cos(\pi b^2) O(x_1; \{x_2\}_3). \quad (3.17)$$

This relation connects $\text{BC}(s_1, 2; s_2, 2)$ and $\text{BC}(s_1, 1; s_2, 3)$ for the correlation of \mathcal{B}_{13} . One can show a general recursion relation between the two-point correlations of $\mathcal{B}_{1, \ell+m-1}$ with different boundary conditions:

$$[m-1]_q O(\{x_1\}_\ell; \{x_2\}_m) = [\ell]_q O(\{x_1\}_{\ell+1}; \{x_2\}_{m-1}). \quad (3.18)$$

Thus, the matrix model predicts the obvious connection between $\text{BC}(s_1, a; s_2, b)$ with $a + b = \ell + m$ for the correlation of $\mathcal{B}_{1, \ell+m-1}$.

One can demonstrate that (3.18) is consistent with the field theoretic result. For the matrix side, one can rescale the matrix $F_k(M; x)$ so that it has the form, $N_k F_k(M; x)$ where N_k is a constant. Then, the two point correlation is rescaled as

$$\frac{1}{N_k N_{\ell+1}} O(\{x_1\}_k; \{x_2\}_{\ell+1}). \quad (3.19)$$

For the field theoretic side, one considers the operator $\Lambda_{1, k+\ell} e^{\beta_{1, k+\ell} \phi} \Phi_{1, k+\ell}$, introducing the normalization constant $\Lambda_{1, k+\ell}$ independent of the BC. The two point correlation of $\mathcal{B}_{1, k+\ell}$ with $\text{BC}(s_1, k; s_2, \ell+1)$ is given as

$$\Lambda_{1, k+\ell}^2 d_L(\beta_{1, k+\ell}, s_1, s_2) d_M(k; \ell+1). \quad (3.20)$$

Here, d_L is the Liouville correlation and d_M is the matter part correlation. Equating two results, Eq. (3.19) and Eq. (3.20) one has

$$O(\{x_1\}_k; \{x_2\}_{\ell+1}) = N_k N_{\ell+1} \Lambda_{1, k+\ell}^2 d_L(\beta_{1, k+\ell}, s_1, s_2) d_M(k; \ell+1). \quad (3.21)$$

On the other hand, Eq. (3.21) should be compatible with Eq. (3.18), which leads to nontrivial relations between N_k 's. $N_k N_{\ell+1} d_M(k, \ell+1) = N_{k+1} N_\ell d_M(k+1, \ell)$. (Note that Liouville part cancels out because the same Liouville $\text{BC}(s_1, s_2)$ and the same operator $\mathcal{B}_{1, k+\ell}$ are used in both sides.) d_M is given in terms of boundary structure constant [23]

$$d_M(k; \ell+1) = c_{(1, k+\ell), (1, k+\ell)}^{(k, \ell+1, k) (11)} \langle 1 \rangle_k = c_{(1, k+\ell), (1, k+\ell)}^{(\ell+1, k, \ell+1) (11)} \langle 1 \rangle_{\ell+1}. \quad (3.22)$$

This simplifies the relation as

$$N_k N_{\ell+1} c_{(1,k+\ell),(1,k+\ell)}^{(k,\ell+1,k)(11)} = N_{k+1} N_\ell c_{(1,k+\ell),(1,k+\ell)}^{(\ell,k+1,\ell)(11)}. \quad (3.23)$$

One can find a consistent solution of N_k 's to Eq. (3.23) which is crucial for the consistency of (3.21). Considering the boundary structure constant is identified with the fusion matrix

$$[23] \ c_{LM}^{(A,B,C)(K)} = F_{B,K} \begin{bmatrix} A, C \\ L, M \end{bmatrix}, \text{ and the fusion matrix is written in a factorized form}$$

when $K = (11)$ [31], $F_{(1i),(11)} \begin{bmatrix} (1k), (1k) \\ (1j), (1j) \end{bmatrix} = \sqrt{\frac{F_j F_k}{F_i}}$ with $F_i = 1/[i]_q$, one has

$$\frac{N_{k+1}}{N_k} \sqrt{\frac{1}{F_k F_{k+1}}} = \frac{N_{\ell+1}}{N_\ell} \sqrt{\frac{1}{F_\ell F_{\ell+1}}}. \quad (3.24)$$

Thus, $N_k (k \geq 3)$ is determined completely from N_1 and N_2 . Finally, putting $k = 1$ in Eq. (3.21), one finds the field normalization $\Lambda_{1,\ell}^2$ in terms of $N_1 N_\ell$. Once N_k 's are determined by the recursion relation (3.24), the identification (3.21) is consistent with the relation (3.18) found in the matrix model.

4 Summary and discussion

We propose a generalized partition function of one-matrix model (2.3) to give boundary correlation numbers of primary fields of $\text{MLG}(2,2p+1)$ on disk. We demonstrate explicitly that the two-flavor vector model correctly reproduces the two-point correlation numbers of boundary primary operator $\mathcal{B}_{1,k}$ ($1 \leq k \leq 2p$) in $\text{MLG}(2,2p+1)$.

A few comments follow. First, one can obtain the general boundary condition by contracting 1-boundary on the disk as given in Eq. (3.13) and Eq. (3.15). One may equally contract ℓ and m -boundaries in Eq. (3.11). In this case one is left with

$$\begin{aligned} {}^{(s_1;1)}[\mathcal{B}_{1\ell}]^{(s_2;\ell)} \otimes {}^{(s_2;\ell)}[\mathcal{B}_{1\ell}]^{(s_3;1)} &= {}^{(s_1;1)}[\mathcal{B}_{11}]^{(s_3;1)} \\ {}^{(s_3;1)}[\mathcal{B}_{1m}]^{(s_4;m)} \otimes {}^{(s_4;m)}[\mathcal{B}_{1m}]^{(s_1;1)} &= {}^{(s_2;1)}[\mathcal{B}_{11}]^{(s_1;1)} \end{aligned} \quad (4.1)$$

since $(1,1)$ -boundary allows only $(1,1)$ operator. This property is seen in the matrix side by contracting ℓ and m -boundaries in Eq. (3.14). Eq. (3.15) is left with the two-point correlation of F_1 's only

$$\left\langle \text{tr} \left(\frac{1}{F_1(M; x_1)} \frac{1}{F_1(M; x_3)} \right) \right\rangle \quad (4.2)$$

and its contribution is \mathcal{B}_{11} with $\text{BC}(s_1, 1; s_2, 1)$.

Second, the non-vanishing power of M ($g \neq 0$) in Eq. (3.15) can be obtained using $C(M)$ in Eq. (2.2) when its off-diagonal component contains the M dependent term up to the power g which is less than any degree of the corresponding diagonal component. Therefore, the M -dependent off-diagonal term produces the two-point correlation of $\mathcal{B}_{1,k}$ of $\text{BC}(s_1, a; s_2, b)$ with $a + b \neq k + 1$. However, it is not clear yet, what kind of mechanism restricts the maximum power g . In addition, fusing rule is not simple for non-vanishing g as shown in Eq. (3.15). This suggests that one needs to find the more detailed description of the M -dependency of the off-diagonal term.

Finally, is the proposal working for three point correlation with 3-flavor vectors? Let us consider $O(x_1; x_2; x_3)$. This produces the 3-point correlation of \mathcal{B}_{11} 's in [30]. On the other hand, the Liouville boundary 3-point correlation $\mathcal{C}_{\beta_1\beta_2\beta_3}^{(s_2, s_3, s_1)}$ of $e^{\beta_i\phi}$'s with $\text{BC}(s_1; s_2; s_3)$ satisfies the difference relation⁶ [24, 25]

$$\begin{aligned} & -\sinh\left\{\frac{\pi b}{2}(s_2 - s_1 + ib - 2i\beta_3)\right\} \sinh\left\{\frac{\pi b}{2}(s_3 - s_1 + ib - 2i\beta_2)\right\} C_{\beta_1\beta_2\beta_3}^{(s_2, s_3, s_1 + ib)} \\ & + \sinh\left\{\frac{\pi b}{2}(s_2 + s_1 + ib + 2i\beta_3)\right\} \sinh\left\{\frac{\pi b}{2}(s_3 + s_1 + ib + 2i\beta_2)\right\} C_{\beta_1\beta_2\beta_3}^{(s_2, s_3, s_1 - ib)} \\ & = \sinh\left\{\frac{\pi b}{2}(s_2 + s_3 + 4ib - 2i\beta_1)\right\} \sinh(\pi b s_1) C_{\beta_1\beta_2\beta_3}^{(s_2 + ib, s_3 + ib, s_1)}. \end{aligned} \quad (4.3)$$

$O(x_1; x_2; x_3)$ obviously satisfies (4.3) with $\beta_1 = \beta_2 = \beta_3 = b$. One can consider a more general form. For example, $O(x_1; x_2; \{x_3\}_2)$ satisfies the difference equation (4.3) with $(\beta_1, \beta_2, \beta_3) = (b, \frac{3b}{2}, \frac{3b}{2})$ and corresponds to $\langle \mathcal{B}_{11} \mathcal{B}_{12} \mathcal{B}_{12} \rangle$ with $\text{BC}(s_2, 2; s_3, 1; s_1, 1)$. Likewise, we expect that the n -point correlation number $O(x_1; \dots; x_n)$ will produce the result $\langle \mathcal{B}_{11} \dots \mathcal{B}_{11} \rangle$ with $\text{BC}(s_1, 1; \dots; s_n, 1)$. Nevertheless, we need more careful check for correlations of other non-trivial operators of $\mathcal{B}_{1,\ell}$'s. In addition, it looks very plausible that the same idea can work for the boundary correlation of multi-matrix model. Further details will be reported elsewhere in the near future.

Note added: After completion of this work authors of [33] let us know that the same model arises in string theory on partially resolved singularities.

⁶We fix a typo in the original equation.

Acknowledgements

We are indebted to A. Belavin and J.-E. Bourguine for the discussion at the initiation of this work and to I. Kostov for encouraging remarks. The work is partially supported by the National Research Foundation of Korea (KNRF) grant funded by the Korea government (MEST) 2005-0049409 (I & R) and R01-2008-000-21026-0 (R).

A Resolvent in the spherical limit

We present an expectation value of the resolvent

$$\omega(z) = \left\langle \text{tr} \frac{1}{z - M} \right\rangle = \left\langle \int_0^\infty dl \text{tr} e^{-l(z-M)} \right\rangle \quad (\text{A.1})$$

in the large- N limit. In the double scaling limit (continuum limit), $M \sim a_1 - a_2 Q$ where a_1, a_2 are constants and $Q = \frac{d^2}{dx^2} + u(x)$ [7, 18, 32]. Thus, one may renormalize $z \rightarrow a_1 + a_2 z$ and put

$$z = u_0 \cosh(\pi b s) \quad (\text{A.2})$$

In the large- N limit, $u(x)$ is given by a maximal real solution to the string equation

$$P(u, \{t_k\}) = u^{p+1} + \sum_{k=0}^{p-1} t_k u^{p-k-1} = 0, \quad (\text{A.3})$$

with $t_{p-1} = x$. The parameter $-t_0$ is proportional to the bulk cosmological constant μ and the other parameters, $\{t_k | 1 \leq k \leq p-1\}$, describe the relevant deviations from the p -critical point in the KdV frame. They are related, through the so-called resonance transformation, to the perturbative coupling constants λ_k which couple to the operators in the minimal gravity. The resonance relation was solved in [17] and $P(u, \{t_k\})$ is given in terms of $\{\lambda_k\}$ as follows,

$$P(u, \{t_k(\lambda_k)\}) = u_0^{p+1} \frac{(p+1)!}{(2p-1)!!} \left(\frac{L_{p+1}(u/u_0) - L_{p-1}(u/u_0)}{2p+1} \right) + \mathcal{O}(\lambda_k), \quad (\text{A.4})$$

where u_0 is a solution to the string equation with $t_k = 0$ for $k \geq 1$, namely, $u_0 = \sqrt{-t_0} \sim \sqrt{\mu}$ and $L_p(x)$ is the Legendre polynomial.

In the large- N limit, we can neglect the commutator $[\frac{d}{dx}, u(x)] = 0$,

$$\omega(z) = \int_0^\infty \frac{dl}{2a_2 \sqrt{\pi l}} \int_{t_0}^\infty dx e^{-l(z-u(x))} \Big|_{\lambda_k=0}$$

$$\begin{aligned}
&= \int_0^\infty \frac{dl}{2a_2\sqrt{\pi\ell}} \int_{u(x=t_0)}^\infty du \frac{dP(u)}{du} e^{-\ell(z-u)} \Big|_{\lambda_k=0} \\
&= u_0^{p+1} \frac{(p+1)!}{2(2p-1)!!a_2} \int_0^\infty \frac{dl}{\sqrt{\pi\ell}} e^{-lz} \int_1^\infty dy L_p(y) e^{-lu_0y}.
\end{aligned} \tag{A.5}$$

If one uses

$$\begin{aligned}
\int_1^\infty dx L_n(x) e^{-lx} &= \sqrt{\frac{2}{\pi\ell}} K_{p+1/2}(l) \\
\int_0^\infty \frac{dl}{2\pi l} e^{-l \cosh(s)} K_{p+\frac{1}{2}}(l) &= \frac{(-1)^{p+2}}{2p+1} \cosh((p+1/2)s)
\end{aligned}$$

for the Macdonald function of a half-integer order $K_{p+1/2}(x)$, one obtains the singular part of the resolvent in the large- N limit,

$$\omega(z) = \alpha u_0^{p+\frac{1}{2}} \cosh\left(\frac{\pi s}{b}\right) \tag{A.6}$$

where $\alpha = \frac{(-1)^{p+2}\sqrt{2}(p+1)!}{a_2(2p+1)!!}$ is independent of s and can be absorbed in the renormalization.

References

- [1] A. Polyakov, Phys. Lett. **B103** (1981) 207.
- [2] V. Knizhnik, A. Polyakov and A. Zamolodchikov, Mod. Phys. Lett. **A3** (1988) 819.
- [3] F. David, Mod. Phys. Lett. **A3** (1988) 1651.
- [4] J. Distler and H. Kawai, Nucl. Phys. **B321** (1989) 509.
- [5] E. Brezin and V. Kazakov, Phys. Lett. **B236** (1990) 144.
- [6] M. Douglas and S. Shenker, Nucl. Phys. **B335** (1990) 635.
- [7] D. Gross and A. Migdal, Phys. Rev. Lett. **64** (1990) 127; Nucl. Phys. **B340** (1990) 333.
- [8] M. Douglas, Phys. Lett. **B238** (1990) 176.
- [9] V. Kazakov, Mod. Phys. Lett **A4** (1989) 2125.
- [10] P. Di Francesco and D. Kutasov, Nucl. Phys. **B375** (1992) 119.
- [11] M. Staudacher, Nucl. Phys. **B336** (1990) 349.

- [12] M. Goulian and M. Li, Phys. Rev. Lett. **66** (1991) 2051.
- [13] A. Belavin and Al. Zamolodchikov, Theor. Math. Phys. **147** (2006) 729.
- [14] P. Ginsparg and G. Moore, “Lectures on 2-D gravity and 2-D string theory (TASI 1992)”, arXiv:hep-th/9304011
- [15] P. Di Francesco, P. Ginsparg and J. Zinn-Justin, Phys. Rep. **254** (1995) 1.
- [16] Al. Zamolodchikov, “Perturbed Conformal Field Theory on Fluctuating Sphere”, arXiv:hep-th/0508044 (2005).
- [17] A.A. Belavin and A. Zamolodchikov, J. Phys. **A42** (2009) 304004.
- [18] G. Moore, N. Seiberg, M. Staudacher, Nucl. Phys. **B362** (1991) 665.
- [19] G. Tarnopolsky, “Five-point Correlation Numbers in One-Matrix Model”, arXiv:0912.4971[hep-th]
- [20] A. Belavin and C. Rim, Phys. Lett. **B687** (2010) 264.
- [21] V. Fateev, A. Zamolodchikov and Al. Zamolodchikov, “Boundary Liouville field theory. 1. Boundary state and boundary two point function”, hep-th/0001012.
- [22] J. L. Cardy, Nucl. Phys. **b 240** (1984) 514; J. L. Cardy and D. C. Lewellen, Phys. Lett. **B259** (1991) 274.
- [23] I. Runkel, Nucl. Phys. **b 549**[FS] (1999) 563.
- [24] B. Ponsot, J. Teschner, Nucl. Phys. **B622** (2002) 309.
- [25] I. Kostov, B. Ponsot, D. Serban, Nucl.Phys. B683 (2004) 309.
- [26] J. E. Bourguine, K. Hosomichi and I. Kostov, Nucl. Phys. B **832**, 462 (2010); J.-E. Bourguine and K. Hosomichi, JHEP **0901**:009 (2009).
- [27] I. Kostov, Nucl.Phys. B658 (2003) 397; J. Stat. Mech. 0708:P08023 (2007).
- [28] J. Jacobsen, H. Saleur, Nucl. Phys. **B788** (2008) 137.
- [29] E. J. Martinec, G. W. Moore and N. Seiberg, Phys. Lett. B **263**, 190 (1991).
- [30] K. Hosomichi, JHEP **0806**:029 (2008).

- [31] G. Moore and N. Seiberg, “Lectures on RCFT”, Physics, Geometry and Topology (Plenum, New York, 1990).
- [32] M. R. Douglas, Phys. Lett. B **238** (1990) 176; T. Banks, M. Douglas, N. Seiberg and S. Shenker, Nucl. Phys. **B238** (1990) 279.
- [33] G. Bonelli, L. Bonora and A. Ricco, Phys. Lett. **B637** (2006) 310.